

On sums of subsets of Chen primes*

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ABSTRACT

In this paper we show that if A is a subset of Chen primes with positive relative density α , then $A + A$ must have positive upper density at least $c\alpha e^{-c' \log(1/\alpha)^{2/3} (\log \log(1/\alpha))^{1/3}}$ in the natural numbers.

1. INTRODUCTION

In 1953, K. Roth [12] proved that any subset of positive integers of positive density contains non-trivial three-term arithmetic progressions. In recent years, Green [5] showed that any subset of primes of relative positive density also has this property. And later Roth's theorem was extended to Chen primes in [6]. Moreover, a celebrated theorem was proved by Green and Tao [7], showing that the primes contain arbitrarily long arithmetic progressions.

The strategy developed by Green and Green-Tao is called “W-trick”. The primes are embedded to a set behaving more “pseudorandom”, meanwhile slight density-increment is gained. For various applications, one can see [2], [4], [9], [10], [11].... In [2], it is proved by Chipeniuk and Hamel that if A is a subset of primes, with positive relative density α_0 , then the set $A + A$ has positive density at least

$$C_1 \alpha_0 e^{-C_2 (\log(1/\alpha_0)^{2/3} (\log \log(1/\alpha_0))^{1/3})}$$

in the natural numbers. This result is not far from best possible due to some examples.

Let \mathcal{P}_c be the set of Chen primes, each of whom is a prime p for which $p + 2$ is either a prime or a product $p_1 p_2$ with $p_1, p_2 > p^{3/11}$, according to Chen[1] and Iwaniec [8]. Chen's famous theorem concludes that there are infinitely many such primes.

Theorem 0. ([8]) Let n be a large integer. Then the number of Chen primes less than n is at least $c_1 n / \log^2 n$, for some absolute constant $c_1 > 0$.

In this paper, we extend the density result to subsets of Chen primes. For any set $S \subseteq \mathbb{N}$, denote

$$\bar{d}(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap [1, n]|}{|[1, n]|}, \quad \bar{d}_{\mathcal{P}_c}(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap \mathcal{P}_c \cap [1, n]|}{|\mathcal{P}_c \cap [1, n]|}.$$

Theorem 1. Let $A \subseteq \mathcal{P}_c$ with positive relative density $\bar{d}_{\mathcal{P}_c}(A) = \alpha_0$. Then

$$\bar{d}(A + A) \geq C_3 \alpha_0 e^{-C_4 (\log(1/\alpha_0)^{2/3} (\log \log(1/\alpha_0))^{1/3})}$$

for some absolute positive constant C_3 and C_4 .

Since $\bar{d}_{\mathcal{P}_c}(A) = \alpha_0$, there exist infinitely many n such that $|A \cap [1, n]| / |\mathcal{P}_c \cap [1, n]| \geq \alpha_0/2$. The previous theorem will follow from a finite version, with $\alpha = \alpha_0/2$.

Theorem 2. Suppose that n is a sufficiently large integer. Let $A \subset \mathcal{P}_c \cap [1, n]$ with $\frac{|A|}{|\mathcal{P}_c \cap [1, n]|} \geq \alpha$. Then

$$|A + A| \geq C_5 \alpha e^{-C_6 \log(1/\alpha)^{2/3} (\log \log(1/\alpha))^{1/3}} n.$$

for some absolute positive constant C_5 and C_6 .

We mainly follow arguments of Chipeniuk-Hamel[2] and combine the envelop sieve function of Green-Tao[6].

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2. NOTATIONS AND PRELIMINARY LEMMAS

For a parameter k , we write $f \ll_k g$ or $f = O_k(g)$ to denote the estimate $f \leq C_k g$ for some positive constant C_k depending only on k . For a set S , $|S|$ and $\#S$ both denote the cardinality of S and the characteristic function $1_S(x)$ takes value 1 for $x \in S$ and 0 otherwise. The sum set $S + S' := \{s + s' : s \in S, s' \in S'\}$. c, c_0, c_1, c_2, \dots are positive absolute constants. We write \mathbb{Z}_N for the cyclic group $\mathbb{Z}/N\mathbb{Z}$ and \mathbb{Z}_N^* for the multiplicative subgroup of integers modulo N . And $[1, N]$ denotes the set $\{1, 2, \dots, N\}$.

Next we introduce Fourier analysis on \mathbb{Z}_N . If $f : \mathbb{Z}_N \rightarrow \mathbb{C}$ is a function and $S \subseteq \mathbb{Z}_N$, we define

$$\mathbb{E}_{x \in S} f(x) := \frac{1}{|S|} \sum_{x \in S} f(x).$$

Let $e_N(x) := e^{2\pi i x/N}$. The Fourier transform and the Fourier inversion are the following

$$\widehat{f}(\xi) = \mathbb{E}_{x \in \mathbb{Z}_N} f(x) e_N(-\xi x), \quad f(x) = \sum_{\xi \in \mathbb{Z}_N} \widehat{f}(\xi) e_N(\xi x).$$

The L^q , L^∞ , l^q , l^∞ -norms are defined to be

$$\|f\|_{L^q} = (\mathbb{E}_{x \in \mathbb{Z}_N} |f(x)|^q)^{1/q}, \quad \|f\|_{L^\infty} = \sup_{x \in \mathbb{Z}_N} |f(x)|.$$

$$\|\widehat{f}\|_{l^q} = \left(\sum_{\xi \in \mathbb{Z}_N} |\widehat{f}(\xi)|^q \right)^{1/q}, \quad \|\widehat{f}\|_{l^\infty} = \sup_{\xi \in \mathbb{Z}_N} |\widehat{f}(\xi)|.$$

Plancherel's equality tells that $\|f\|_{L^2} = \|\widehat{f}\|_{l^2}$. We also write

$$f * g(x) = \mathbb{E}_{y \in \mathbb{Z}_N} f(x - y) g(y)$$

for convolution. A basic identity for convolution is $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. For non-negative valued function f and g , it obeys that

$$\|f * g\|_{L^1} = \|f\|_{L^1} \cdot \|g\|_{L^1}. \quad (1)$$

Hausdorff-Young inequality shows that

$$\|\widehat{f}\|_{l^{q'}} \leq \|f\|_{L^q},$$

where $1 \leq q \leq 2$ and $1/q + 1/q' = 1$. Hölder's inequality says that

$$\|fg\|_{L^r} \leq \|f\|_{L^q} \|g\|_{L^{q'}}$$

whenever $0 < q, q', r \leq \infty$ are such that $1/q + 1/q' = 1/r$. And Young inequality tells that

$$\|f * g\|_{L^r} \leq \|f\|_{L^q} \|g\|_{L^{q'}}$$

for $1 \leq q, q', r \leq \infty$ and $1/q + 1/q' = 1/r + 1$. (See [13] for details.)

Throughout this paper, $A \subseteq \mathcal{P}_c \cap [1, n]$ with $|A|/|\mathcal{P}_c \cap [1, n]| \geq \alpha$. And p always denotes a prime. Let $\lambda, \varepsilon, \delta$ be small parameters and $t \gg 1$ be a very large real number to be specified later.

Write $W := \prod_{3 \leq p \leq t} p$. For b with $(b, W) = (b + 2, W) = 1$, write $F^{(b)}(x) = (Wx + b)(Wx + b + 2)$, and let $R = \lfloor N^{1/20} \rfloor$. In [6], Green and Tao constructed an enveloping sieve function $\beta_R^{(b)}$ with the property (See [6] Proposition 3.1)

$$\beta_R^{(b)}(x) \gg \mathfrak{S}_{F^{(b)}}^{-1} \log^2 R \cdot 1_{X_{R!}^{(b)}}(x) \quad (2)$$

(One can check that the constant does not depend on b .) with

$$\mathfrak{S}_{F^{(b)}} = \prod_p \frac{\gamma^{(b)}(p)}{\left(1 - \frac{1}{p}\right)^2}, \quad \gamma^{(b)}(p) = \frac{1}{p} |\{n \in \mathbb{Z}/p\mathbb{Z} : (p, F^{(b)}(n)) = 1\}|,$$

and

$$X_{R!}^{(b)} = \left\{ n \in \mathbb{Z} : (d, F^{(b)}(n)) = 1 \text{ for all } 1 \leq d \leq R \right\}.$$

By computation (or See [6] (6.3)),

$$\log^2 t \ll \mathfrak{S}_{F^{(b)}} \ll \log^2 t. \quad (3)$$

Restrict $\beta_R^{(b)}$ to the set $[1, N]$, which we identify with \mathbb{Z}_N , and let $\nu^{(b)} : \mathbb{Z}_N \rightarrow \mathbb{R}^+$ be the resulting function. We have the following

Lemma 1. ([6], Lemma 6.1) For $\xi \in \mathbb{Z}_N$, we have

$$\widehat{\nu^{(b)}}(\xi) = \delta_{\xi,0} + O(t^{-1/2}),$$

where $\delta_{\xi,0}$ is the *Kronecker* delta function. Also the constant does not depend on b .

Lemma 2. ([6], Prop 4.2, Lemma 3R.1) Let $q > 2$ be a real number and $\{a_x\}$ be an arbitrary sequence of complex numbers. Suppose that $1 \ll R \ll N^{1/10}$ and $F^{(b)}, \beta_R^{(b)}$ are defined above. Then we have

$$\left(\sum_{\xi \in \mathbb{Z}_N} \left| \mathbb{E}_{1 \leq x \leq N} a_x \beta_R^{(b)}(x) e_N(-\xi x) \right|^q \right)^{1/q} \ll_q \left(\mathbb{E}_{1 \leq x \leq N} |a_x|^2 \beta_R^{(b)}(x) \right)^{1/2}$$

and

$$\mathbb{E}_{1 \leq x \leq N} \beta_R^{(b)}(x) \ll 1.$$

3. W-TRICK TO CHEN PRIMES

Denote $A_n := A \cap (\sqrt{n}, n]$. And let $\Phi_W := \{b \in \mathbb{Z}_W : (b, W) = (b+2, W) = 1\}$. Observe that

$$\varphi_W := |\Phi_W| = W \prod_{3 \leq p \leq t} \left(1 - \frac{2}{p}\right) \ll \frac{W}{\log^2 t}.$$

Choose a prime $N \in (\frac{2n}{W}, \frac{4n}{W}]$. For $b \in \Phi_W$, let

$$A_n^{(b)} := \{x \in A_n : x \equiv b \pmod{W}\}, \quad A_N^{(b)} := \{x \leq N : Wx + b \in A_n^{(b)}\}.$$

The choice of N ensures that $A_n^{(b)} + A_n^{(b)}$ in \mathbb{Z} can be identified with $A_N^{(b)} + A_N^{(b)}$ in \mathbb{Z}_N . Noting that $A_N^{(b)} \subseteq X_{R!}^{(b)}$. Combining (2) and (3), we can define $f^{(b)} : \mathbb{Z}_N \rightarrow \mathbb{R}^+$ by

$$f^{(b)}(x) = c \frac{\log^2 N}{\log^2 t} 1_{A_N^{(b)}}(x),$$

with $c > 0$ chosen sufficiently small such that $f^{(b)}(x) \leq \nu^{(b)}(x)$. (Noting that the choice of c does not depend on b .)

Now follow [5] and [6], we decompose $f^{(b)}$ into one *anti-uniform* component and one *uniform* component. Denote

$$R^{(b)} = \{\xi \in \mathbb{Z}_N : |\widehat{f^{(b)}}(\xi)| \geq \delta\}, \quad B^{(b)} = \{x \in \mathbb{Z}_N : \sup_{\xi \in R^{(b)}} |1 - e_N(\xi x)| \leq \varepsilon\}.$$

Let $\beta^{(b)}(x) = \frac{N}{|B^{(b)}|} 1_{B^{(b)}}(x)$ and

$$f_1^{(b)} = f^{(b)} * \beta^{(b)} * \beta^{(b)}, \quad f_2^{(b)} = f^{(b)} - f_1^{(b)}.$$

Lemma 3. The functions $f^{(b)}, f_1^{(b)}, f_2^{(b)}$ defined above have the following properties:

- (i) $\|f_1^{(b)}\|_{L^1} = \|f^{(b)}\|_{L^1} \leq 1 + O(t^{-1/2})$.
- (ii) $\left\| \widehat{f_2^{(b)}} \right\|_{l^\infty} \ll \delta + \varepsilon \cdot (1 + O(t^{-1/2}))$.
- (iii) $\left\| \widehat{f_1^{(b)}} \right\|_{l^{2+\lambda}}, \left\| \widehat{f_2^{(b)}} \right\|_{l^{2+\lambda}} \ll \left\| \widehat{f^{(b)}} \right\|_{l^{2+\lambda}} \ll_\lambda 1$.
- (iv) $\left\| f_1^{(b)} \right\|_{L^\infty} \leq 1 + O(t^{-1/2}) + O_\lambda(\varepsilon^{-c_\lambda} \delta^{-(2+\lambda)} t^{-1/2})$.

Proof: The proofs follow as in [6]. We reiterate them here in order to specify ε and δ .

(i) One can deduce that $\|\beta^{(b)}\|_{L^1} = 1$. Then by (1) and Lemma 1, we have

$$\|f_1^{(b)}\|_{L^1} = \|f^{(b)}\|_{L^1} \leq \|\nu^{(b)}\|_{L^1} = |\widehat{\nu^{(b)}}(0)| = 1 + O(t^{-1/2}).$$

(ii) Noting that

$$\widehat{f_2^{(b)}}(\xi) = \widehat{f^{(b)}}(\xi) - \widehat{f_1^{(b)}}(\xi) = \widehat{f^{(b)}}(\xi)(1 - \widehat{\beta^{(b)}}(\xi)^2).$$

For $\xi \in R^{(b)}$, we have

$$|1 - \widehat{\beta^{(b)}}(\xi)| = |\mathbb{E}_{x \in B^{(b)}}(1 - e_N(-\xi x))| \leq \mathbb{E}_{x \in B^{(b)}}|1 - e_N(-\xi x)| \leq \varepsilon.$$

By triangle inequality, $|1 - \widehat{\beta^{(b)}}(\xi)^2| \leq 2\varepsilon$. Also we have

$$\|\widehat{f^{(b)}}\|_{l^\infty} \leq \|f^{(b)}\|_{L^1} \leq \|\nu^{(b)}\|_{L^1} = 1 + O(t^{-1/2})$$

by Hausdorff-Young inequality and Lemma 1. Hence

$$|\widehat{f_2^{(b)}}(\xi)| \ll \varepsilon \cdot (1 + O(t^{-1/2})).$$

And for $\xi \notin R^{(b)}$, we have

$$|\widehat{f_2^{(b)}}(\xi)| \ll |\widehat{f^{(b)}}(\xi)| \ll \delta.$$

(iii) For arbitrary $\lambda > 0$, taking $a_x = f^{(b)}(x)/\beta_R^{(b)}(x)$ and $q = 2 + \lambda$, Lemma 2 tells

$$\|\widehat{f^{(b)}}\|_{l^{2+\lambda}} \ll_\lambda 1.$$

Since $\|\widehat{\beta^{(b)}}\|_{l^\infty} \leq \|\beta^{(b)}\|_{L^1} = 1$, we have

$$\left| \widehat{f_1^{(b)}}(\xi) \right| = |\widehat{f^{(b)}}(\xi)| |\widehat{\beta^{(b)}}(\xi)|^2 \leq |\widehat{f^{(b)}}(\xi)|,$$

and

$$\left| \widehat{f_2^{(b)}}(\xi) \right| \leq \left| \widehat{f^{(b)}}(\xi) \right| + \left| \widehat{f_1^{(b)}}(\xi) \right| \ll \left| \widehat{f^{(b)}}(\xi) \right|.$$

Then (iii) follows.

(iv)

$$\begin{aligned} |f_1^{(b)}(x)| &= |f^{(b)} * \beta^{(b)} * \beta^{(b)}(x)| \\ &\leq |\nu^{(b)} * \beta^{(b)} * \beta^{(b)}(x)| \\ &= \left| \sum_{\xi \in \mathbb{Z}_N} \widehat{\nu^{(b)}}(\xi) \widehat{\beta^{(b)}}(\xi)^2 e_N(\xi x) \right| \\ &\leq |\widehat{\nu^{(b)}}(0)| |\widehat{\beta^{(b)}}(0)|^2 + \sup_{\xi \neq 0} |\widehat{\nu^{(b)}}(\xi)| \sum_{\xi} |\widehat{\beta^{(b)}}(\xi)|^2 \\ &\leq (1 + O(t^{-1/2})) + O(t^{-1/2}) \cdot \frac{N}{|B^{(b)}|}. \end{aligned}$$

In the last inequality we have used Plancherel's equality to get

$$\sum_{\xi} |\widehat{\beta^{(b)}}(\xi)|^2 = \mathbb{E}_{x \in \mathbb{Z}_N} |\beta^{(b)}(x)|^2 = \frac{N}{|B^{(b)}|}.$$

By pigeonhole principle (or see [3], Lemma 3) we have $|B^{(b)}| \gg (\varepsilon/2\pi)^{|R^{(b)}|} N$. And from (iii), there exists some positive integer c_λ such that

$$c_\lambda \geq \|\widehat{f^{(b)}}\|_{l^{2+\lambda}} \geq \sum_{\xi \in R} |\widehat{f}(\xi)|^{2+\lambda} \geq |R^{(b)}| \cdot \delta^{2+\lambda},$$

which implies $|R^{(b)}| \leq c_\lambda \delta^{-(2+\lambda)}$. Hence we get

$$0 \leq f_1^{(b)}(x) \leq 1 + O(t^{-1/2}) + O_\lambda(\varepsilon^{-c_\lambda \delta^{-(2+\lambda)}} t^{-1/2}).$$

□

Now for $b \in \Phi_W$, define

$$\delta_b := \frac{|A_n^{(b)}|}{\frac{c_1 n}{\log^2 n \cdot \varphi_W}}.$$

By (i) of Lemma 3, the definition of $f^{(b)}$ and recalling that $N \in (\frac{2n}{W}, \frac{4n}{W}]$, we can get

$$\delta_b \ll \|f^{(b)}\|_{L^1} \ll \delta_b \ll 1, \quad (4)$$

when $t \gg 1$ is sufficient large. However, $\delta_b \ll 1$ is equivalent to

$$|A_n^{(b)}| \ll \frac{n}{\log^2 n \cdot \varphi_W}. \quad (5)$$

Lemma 4. The convolution of functions $f_1^{(b)}$ and $f_2^{(b)}$ defined above have following properties:

- (i) $\delta_{b_1} \delta_{b_2} \ll \|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^1} \ll \delta_{b_1} \delta_{b_2}$.
- (ii) $\|f_1^{(b_1)} * f_2^{(b_2)}\|_{L^2} \ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2})))^{1-\lambda/2}$, $\|f_2^{(b_1)} * f_2^{(b_2)}\|_{L^2} \ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2})))^{1-\lambda/2}$.
- (iii) $\|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^\infty} \ll \min\{\delta_{b_1}, \delta_{b_2}\}(1 + O(t^{-1/2}) + O_\lambda(\varepsilon^{-c_\lambda} \delta^{-(2+\lambda)} t^{-1/2}))$.

Proof: (i) By (1) and (4), we have

$$\|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^1} = \|f_1^{(b_1)}\|_{L^1} \|f_1^{(b_2)}\|_{L^1} = \|f^{(b_1)}\|_{L^1} \|f^{(b_2)}\|_{L^1} \gg \delta_{b_1} \delta_{b_2}.$$

(ii) Using Plancherel's equality, Hölder's inequality, and together with Lemma 3, we can obtain

$$\begin{aligned} \|f_1^{(b_1)} * f_2^{(b_2)}\|_{L^2}^2 &= \left\| \widehat{f_1^{(b_1)}} \widehat{f_2^{(b_2)}} \right\|_{l^2}^2 = \left\| \widehat{f_1^{(b_1)}}^2 \widehat{f_2^{(b_2)}}^2 \right\|_{l^1} \\ &\leq \left\| \widehat{f_2^{(b_2)}}^{2-\lambda} \right\|_{l^\infty} \left\| \widehat{f_1^{(b_1)}}^2 \widehat{f_2^{(b_2)}}^\lambda \right\|_{L^1} \\ &\leq \left\| \widehat{f_2^{(b_2)}} \right\|_{l^\infty}^{2-\lambda} \cdot \left\| \widehat{f_1^{(b_1)}} \right\|_{l^{(2+\lambda)/2}}^2 \cdot \left\| \widehat{f_2^{(b_2)}} \right\|_{l^{(2+\lambda)/\lambda}}^\lambda \\ &\leq \left\| \widehat{f_2^{(b_2)}} \right\|_{l^\infty}^{2-\lambda} \cdot \left\| \widehat{f_1^{(b_1)}} \right\|_{l^{2+\lambda}}^2 \cdot \left\| \widehat{f_2^{(b_2)}} \right\|_{l^{2+\lambda}}^\lambda \\ &\ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2})))^{2-\lambda}. \end{aligned}$$

(iii) With the Young inequality and Lemma 3, it follows that

$$\begin{aligned} \|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^\infty} &\leq \|f_1^{(b_1)}\|_{L^1} \cdot \|f_1^{(b_2)}\|_{L^\infty} \\ &\ll \delta_{b_1} (1 + O(t^{-1/2}) + O_\lambda(\varepsilon^{-c_\lambda} \delta^{-(2+\lambda)} t^{-1/2})). \end{aligned}$$

Similarly, we have

$$\|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^\infty} \ll \delta_{b_2} (1 + O(t^{-1/2}) + O_\lambda(\varepsilon^{-c_\lambda} \delta^{-(2+\lambda)} t^{-1/2})).$$

□

For any $b_1, b_2 \in \Phi_W$, define $T^{(b_1, b_2)} = \{x \in \mathbb{Z}_N : f^{(b_1)} * f^{(b_2)}(x) > 0\}$.

Proposition 1. For any $b_1, b_2 \in \Phi_W$, we have

$$|T^{(b_1, b_2)}| \gg (\delta_{b_1} + \delta_{b_2})N,$$

provided that $\varepsilon, \delta \ll \min\{\delta_{b_1}, \delta_{b_2}\}^{\frac{5}{2-\lambda}}$ and $\log t \gg_\lambda \left(\frac{1}{\delta}\right)^{2+\lambda} \log \frac{1}{\varepsilon}$.

Proof: Without loss of generality, we suppose that $\delta_{b_1} \leq \delta_{b_2}$. And suppose $\|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^1} = c_2 \delta_{b_1} \delta_{b_2}$ with $1 \ll c_2 \ll 1$ (Recalling Lemma 4(i)). Let

$$T_{1,1} = \{x \in \mathbb{Z}_N : f_1^{(b_1)} * f_1^{(b_2)}(x) > c_2 \delta_{b_1} \delta_{b_2} / 2\},$$

and

$$T_{i,j} = \{x \in \mathbb{Z}_N : |f_i^{(b_1)} * f_j^{(b_2)}(x)| > c_2 \delta_{b_1} \delta_{b_2} / 20\}.$$

for $(i, j) = (1, 2), (2, 1)$ or $(2, 2)$. Then

$$T^{(b_1, b_2)} \supseteq T_{1,1} \cap (T_{1,2} \cup T_{2,1} \cup T_{2,2})^c.$$

So

$$|T^{(b_1, b_2)}| \geq |T_{1,1}| - |T_{1,2}| - |T_{2,1}| - |T_{2,2}|. \quad (6)$$

Combining Lemma 4, we can get

$$\begin{aligned} c_2 \delta_{b_1} \delta_{b_2} &= \|f_1^{(b_1)} * f_1^{(b_2)}\|_{L^1} = \mathbb{E}_{x \in \mathbb{Z}_N} f_1^{(b_1)} * f_1^{(b_2)}(x) \\ &\leq \frac{1}{N} \left(|T_{1,1}| \cdot \delta_{b_1} (1 + O(t^{-1/2})) + O_\lambda(\varepsilon^{-c_\lambda \delta^{-(2+\lambda)}} t^{-1/2}) + N \cdot c_2 \delta_{b_1} \delta_{b_2} / 2 \right). \end{aligned}$$

We conclude that

$$|T_{1,1}| \gg \delta_{b_2} (1 + O(t^{-1/2})) + O_\lambda(\varepsilon^{-c_\lambda \delta^{-(2+\lambda)}} t^{-1/2})^{-1} N. \quad (7)$$

For $(i, j) = (1, 2), (2, 1)$ or $(2, 2)$, it appears that

$$\begin{aligned} (\delta + \varepsilon \cdot (1 + O(t^{-1/2})))^{2-\lambda} &\gg \|f_i^{(b_1)} * f_j^{(b_2)}\|_{L^2}^2 \geq \frac{1}{N} |T_{i,j}| \cdot (c_2 \delta_{b_1} \delta_{b_2} / 20)^2. \\ |T_{i,j}| &\ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2})))^{2-\lambda} (\delta_{b_1} \delta_{b_2})^{-2} N. \end{aligned} \quad (8)$$

Choose $\varepsilon = \delta \ll \min\{\delta_{b_1}, \delta_{b_2}\}^{\frac{5}{2-\lambda}}, t \gg 1$ and $\log t \gg c_\lambda \left(\frac{1}{\delta}\right)^{2+\lambda} \log \frac{1}{\varepsilon}$ such that

$$|T_{1,1}| > 4 \max\{|T_{1,2}|, |T_{2,1}|, |T_{2,2}|\},$$

then Proposition 1 follows from (6), (7) and (8). \square

4. SUMS OF SUBSETS OF CHEN PRIMES

Noting that $A \subseteq \mathcal{P}_c \cap [1, n]$ with $|A| \geq \alpha |\mathcal{P}_c \cap [1, n]|$ and $A_n = A \cap (\sqrt{n}, n]$. By Theorem 0, we can assert that

$$|A_n| \geq \frac{\alpha c_1 n}{2 \log^2 n}. \quad (9)$$

To pick out the $A_n^{(b)}$'s with 'many' elements, define

$$G := \left\{ b \in \Phi_W : \delta_b \geq \frac{\alpha}{4} \right\}.$$

Recall (5), i.e.

$$|A_n^{(b)}| \ll \frac{n}{\log^2 n \cdot \varphi_W}.$$

Since

$$\frac{\alpha c_1 n}{2 \log^2 n} \leq \sum_{b \in \Phi_W} |A_n^{(b)}| \leq |G| O(1) \frac{n}{\log^2 n \cdot \varphi_W} + \varphi_W \cdot \frac{\alpha}{4} \cdot \frac{c_1 n}{\log^2 n \cdot \varphi_W},$$

we conclude that

$$|G| \gg \alpha \varphi_W.$$

Proposition 1 tells that

$$|A_n^{(b_1)} + A_n^{(b_2)}| = |A_N^{(b_1)} + A_N^{(b_2)}| \geq |T^{(b_1, b_2)}| \gg (\delta_{b_1} + \delta_{b_2}) N \gg (\delta_{b_1} + \delta_{b_2}) n / W,$$

provided that we set $\varepsilon = \delta = c_3 \alpha^{\frac{5}{2-\lambda}}$ and $\log t = c_4 c_\lambda \alpha^{-\frac{5(2+\lambda)}{2-\lambda}} \log \alpha^{-1}$.

Denote $\Delta_x = \max_{\substack{(b_1, b_2) \in G \times G \\ b_1 + b_2 = x}} (\delta_{b_1} + \delta_{b_2})$. we have

$$|A_n + A_n| \geq \sum_{x \in G+G} \max_{\substack{(b_1, b_2) \in G \times G \\ b_1 + b_2 = x}} |A_n^{(b_1)} + A_n^{(b_2)}| \gg \sum_{x \in G+G} \Delta_x \cdot n / W. \quad (10)$$

By (9),

$$\sum_{b \in \Phi_W} \delta_b \geq \alpha \varphi_W / 2.$$

Then we have

$$\sum_{b \in G} \delta_b \geq \alpha \varphi_W / 4.$$

$$\sum_{(b_1, b_2) \in G \times G} (\delta_{b_1} + \delta_{b_2}) \gg \alpha \varphi_W |G|.$$

For $B \subseteq \mathbb{Z}_W$ and $x \in \mathbb{Z}_W$, denote $r_B(x) = \#\{(b_1, b_2) \in G \times G : b_1 + b_2 = x\}$.

Lemma 5. Suppose $W \in \mathbb{Z}^+$ is a sufficiently large squarefree integer. Let $\alpha > 0$ and k sufficiently large. And let $B \subseteq \mathbb{Z}_W$ satisfy $|B| \geq \alpha \varphi_W$. Then

$$\sum_{x \in \mathbb{Z}_W} r_B(x)^k \leq \frac{e^{\tilde{C} k^3 \log k}}{\alpha^2} \frac{|G|^k \varphi_W^k}{W^{k-1}}$$

for some absolute constant $\tilde{C} > 0$.

This lemma is an analog of Proposition 14 of Chipeniuk and Hamel[2]. It can be extended to any subsets of ‘sieve-type’ without much modification. We put the long proof in the appendix.

By Lemma 5, we conclude

$$\sum_{x \in G+G} r_G(x)^k \leq \frac{e^{O(k^3 \log k)}}{\alpha^2} \frac{|G|^k \varphi_W^k}{W^{k-1}}.$$

By Hölder’s inequality,

$$\begin{aligned} \alpha \varphi_W |G| &\ll \sum_{x \in G+G} r_G(x) \left(\frac{1}{r_G(x)} \sum_{(b_1, b_2): b_1 + b_2 = x} (\delta_{b_1} + \delta_{b_2}) \right) \\ &\leq \sum_{x \in G+G} r_G(x) \cdot \Delta_x \\ &\leq \left(\sum_{x \in G+G} r_G(x)^k \right)^{1/k} \left(\sum_{x \in G+G} \Delta_x^{k/(k-1)} \right)^{(k-1)/k} \\ &\ll \frac{e^{O(k^2 \log k)}}{\alpha^{2/k}} \frac{|G| \varphi_W}{W^{(k-1)/k}} \cdot \left(\sum_{x \in G+G} \Delta_x^{k/(k-1)} \right)^{(k-1)/k}. \end{aligned}$$

Noting that $\Delta_x \leq 2$. Calculation shows that

$$\begin{aligned} \sum_{x \in G+G} \Delta_x &\gg \sum_{x \in G+G} \Delta_x^{k/(k-1)} \\ &\gg \alpha^{1 + \frac{3}{k-1}} e^{-O(\frac{k^3 \log k}{k-1})} W. \end{aligned} \tag{11}$$

Combining (10) and (11), yields

$$|A_n + A_n| \gg \alpha e^{-O(\frac{k^3 \log k}{k-1}) + \frac{3 \log \alpha}{k-1}} n.$$

If α is small enough, it can be deduced that

$$|A + A| \geq |A_n + A_n| \gg \alpha e^{-O((\log \alpha^{-1})^{2/3} (\log \log \alpha^{-1})^{1/3})} n$$

by taking $k = \lfloor (\log \alpha^{-1} / \log \log \alpha^{-1})^{1/3} \rfloor$. For α is not small, Theorem 2 can also follow by partition B into the union of smaller subsets such that the above argument can be applied. (See [2])

□

5. APPENDIX: ADDITION IN Φ_W

Proof of Lemma 5: Let

$$\begin{aligned} R(x) &:= \#\{(b, r) \in B \times \Phi_W : b + r = x\} \\ &= \#\{b \in B : (b - x)(b - x + 2) \not\equiv 0 \pmod{p} \text{ for all } p|W\}. \end{aligned}$$

Put

$$\begin{aligned} X_d &:= \{x \in [0, W - 1] : (x, W) = d\} \\ &= \{x \in [0, W - 1] : x = dl \text{ for some } l \in [0, W/d - 1] \text{ with } (l, W/d) = 1\}. \end{aligned}$$

We have

$$\begin{aligned} S &= \sum_{x \in \mathbb{Z}_W} r_B(x)^k \\ &\leq \sum_{x \in \mathbb{Z}_W} R(x)^k \\ &= \sum_{d|W} \sum_{x \in X_d} R(x)^k \\ &= \sum_{d|W} \sum_{x \in X_d} \#\{b \in B : (b - x)(b - x + 2) \not\equiv 0 \pmod{p} \text{ for all } p|W/d\}^k \\ &\leq \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \#\{x \in X_d : (b_i - x)(b_i - x + 2) \not\equiv 0 \pmod{p} \text{ for all } p|W/d \text{ and } 1 \leq i \leq k\} \\ &\leq \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \#\{l \in [0, W/d - 1] : (l, W/d) = 1, \\ &\quad (b_i d^{-1} - l)(b_i d^{-1} - l + 2d^{-1}) \not\equiv 0 \pmod{p} \text{ for all } p|W/d \text{ and } 1 \leq i \leq k\} \\ &\leq \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \#\{l \in [0, W/d - 1] : l \not\equiv 0 \pmod{p} \text{ for all } p|W/d, \\ &\quad (b_i d^{-1} - l)(b_i d^{-1} - l + 2d^{-1}) \not\equiv 0 \pmod{p} \text{ for all } p|W/d \text{ and } 1 \leq i \leq k\} \\ &= \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \prod_{p|W/d} (p - r_p(b_1, \dots, b_k) - 1) \\ &= \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \frac{W}{d} \prod_{p|W/d} \left(1 - \frac{r_p(b_1, \dots, b_k) + 1}{p}\right), \end{aligned}$$

where

$$r_p(b_1, \dots, b_k) = \#\{s \in [0, p - 1] : (b_i d^{-1} - s)(b_i d^{-1} - s + 2d^{-1}) \equiv 0 \pmod{p} \text{ for some } 1 \leq i \leq k\}.$$

Now we fix a $d|W$.

Claim. For $(b_1, \dots, b_k) \in B^k$, let

$$f(b_1, \dots, b_k) = \sum_{\substack{p|W \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \frac{1}{p}.$$

and

$$K = \{(b_1, \dots, b_k) \in B^k : f(b_1, \dots, b_k) \geq \beta\}.$$

Then there exists an absolute constant $c > 0$ such that

$$|K| \leq k^2 2^{-\exp(\beta/c k^2)} |B|^{k-2} \varphi_W^2.$$

holds uniformly for every $\beta > 0$.

Proof: Given $(b_1, \dots, b_k) \in K$, we have

$$\begin{aligned}
\beta &\leq \sum_{\substack{p|W \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \frac{1}{p} \\
&\leq \sum_{\substack{p|W \\ p|(b_i - b_j)(b_i - b_j - 2) \text{ for some } i \neq j}} \frac{1}{p} \\
&\leq \sum_{\substack{\{i, j\} \subseteq \{1, \dots, k\} \\ i \neq j}} \sum_{\substack{p|W \\ p|(b_i - b_j)(b_i - b_j - 2)}} \frac{1}{p}.
\end{aligned}$$

By the pigeon hole principle, there exists some pair $\{i, j\}$ with $i \neq j$ such that

$$\frac{\beta}{k(k-1)/2} \leq \sum_{\substack{p|W \\ p|(b_i - b_j)(b_i - b_j - 2)}} \frac{1}{p}.$$

Since each $(b_1, \dots, b_k) \in K$ contribute at least one such pair $\{i, j\}$ and a given $\{b_i, b_j\}$ comes from at most $k^2|B|^{k-2}$ k -tuples, hence we can conclude

$$\frac{|K|}{k^2|B|^{k-2}} 2^l \beta^l k^{-l} (k-1)^{-l} \leq \sum_{\substack{\{b, c\} \subseteq B \\ b \neq c}} \left(\sum_{\substack{p|W \\ p|(b-c)(b-c-2)}} \frac{1}{p} \right)^l. \quad (12)$$

Furthermore, we have

$$\begin{aligned}
&\sum_{\substack{\{b, c\} \subseteq B \\ b \neq c}} \left(\sum_{\substack{p|W \\ p|(b-c)(b-c-2)}} \frac{1}{p} \right)^l \\
&\leq \sum_{(b, c) \in B^2} \left(\sum_{\substack{p|W \\ p|(b-c)(b-c-2)}} \frac{1}{p} \right)^l \\
&= \sum_{p_1, \dots, p_l | W} \frac{1}{p_1 \dots p_l} \sum_{\substack{(b, c) \in B^2 \\ (b-c)(b-c-2) \equiv 0 \pmod{lcm[p_1, \dots, p_l]}}} 1
\end{aligned}$$

Write $p_0 = \max_{1 \leq i \leq l} p_i$ for fixed p_1, \dots, p_l , one can deduce that

$$\begin{aligned}
&\sum_{\substack{(b, c) \in B^2 \\ (b-c)(b-c-2) \equiv 0 \pmod{lcm[p_1, \dots, p_l]}}} 1 \leq \sum_{b \in \Phi_W} \sum_{\substack{c \in \Phi_W \\ (b-c)(b-c-2) \equiv 0 \pmod{p_0}}} 1 \\
&\leq 2 \sum_{b \in \Phi_W} \max_{a \in \mathbb{Z}_W} \left\{ \sum_{\substack{c \in \varphi_W \\ c \equiv a \pmod{p_0}}} 1 \right\} \leq 2\varphi_W \max_{a \in \mathbb{Z}_W} \left\{ \sum_{\substack{c \equiv a \pmod{p_0} \\ c \not\equiv 0, -2 \pmod{p} \text{ for all } p|W/p_0}} 1 \right\} \\
&\leq 2\varphi_W \varphi_{W/p_0} \leq \frac{2\varphi_W^2}{p_0 - 2} \leq \frac{2\varphi_W^2}{\prod_{1 \leq i \leq l} (p_i - 2)^{1/l}}.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{\substack{\{b,c\} \subseteq B \\ b \neq c}} \left(\sum_{\substack{p|W \\ p|(b-c)(b-c-2)}} \frac{1}{p} \right)^l \\
& \leq 2\varphi_W^2 \sum_{p_1, \dots, p_l | W} \frac{1}{\prod_{1 \leq i \leq l} p_i (p_i - 2)^{1/l}} \\
& = 2\varphi_W^2 \left(\sum_{p|W} \frac{1}{p(p-2)^{1/l}} \right)^l \\
& \leq 2\varphi_W^2 \left(\sum_{p \leq l^l} \frac{1}{p} + \sum_{n \geq l^l} \frac{1}{n(n-2)^{1/l}} \right)^l \\
& \leq \varphi_W^2 (c \log l)^l
\end{aligned}$$

for some absolute constant $c > 0$. Combining (12) and the above formula, gives

$$|K| \leq 2^{-l} \beta^{-l} k^{l+2} (k-1)^l c^l (\log l)^l |B|^{k-2} \varphi_W^2.$$

The Claim follows by taking $l = \exp(\beta/c k^2)$.

□

Writing

$$W_1 = \prod_{\substack{p|W \\ p \leq 5k}} p, W_2 = \prod_{\substack{p|W \\ p > 5k}} p.$$

$$d_1 = \prod_{\substack{p|d \\ p \leq 5k}} p, d_2 = \prod_{\substack{p|d \\ p > 5k}} p.$$

The estimate below will be useful later.

$$\begin{aligned}
& \log \left(\prod_{\substack{p|W_2/d_2 \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \left(1 - \frac{2k+1}{p} \right)^{-1} \right) \\
& = - \sum_{\substack{p|W_2/d_2 \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \log \left(1 - \frac{2k+1}{p} \right) \\
& = \sum_{\substack{p|W_2/d_2 \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \frac{2k+1}{p} \sum_{t=0}^{\infty} \frac{1}{t+1} \left(\frac{2k+1}{p} \right)^t \\
& \leq 2(2k+1) \sum_{\substack{p|W_2/d_2 \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \frac{1}{p} \\
& \leq 2(2k+1) f(b_1, \dots, b_k).
\end{aligned}$$

The last step is resulted from the fact that $\frac{1}{t+1} \left(\frac{2k+1}{p} \right)^t \leq \frac{1}{2^t}$ for $t \geq 1$ and $p > 5k$. Noting that $r_p(b_1, \dots, b_k) \geq 1$. We continue to estimate S .

$$\begin{aligned}
S &\leq \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \frac{W}{d} \prod_{p|W/d} \left(1 - \frac{r_p(b_1, \dots, b_k) + 1}{p} \right) \\
&\leq \sum_{d|W} \sum_{b_1, \dots, b_k \in B} \frac{W_1}{d_1} \prod_{p|W_1/d_1} \left(1 - \frac{2}{p} \right) \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{r_p(b_1, \dots, b_k) + 1}{p} \right) \\
&= \sum_{d|W} \varphi_{W_1/d_1} \sum_{b_1, \dots, b_k \in B} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{r_p(b_1, \dots, b_k) + 1}{p} \right) \\
&\leq \sum_{d|W} \varphi_{W_1/d_1} \sum_{j=-\infty}^{\infty} \sum_{\substack{b_1, \dots, b_k \in B \\ f(b_1, \dots, b_k) \in [2^j, 2^{j+1})}} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{2k+1}{p} \right) \prod_{\substack{p|W_2/d_2 \\ r_p(b_1, \dots, b_k) \leq 2k-1}} \left(1 - \frac{2k+1}{p} \right)^{-1} \\
&\leq \sum_{d|W} \varphi_{W_1/d_1} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{2k+1}{p} \right) \sum_{j=-\infty}^{\infty} \sum_{\substack{b_1, \dots, b_k \in B \\ f(b_1, \dots, b_k) \in [2^j, 2^{j+1})}} e^{(2k+1)2^{j+2}} \\
&\leq \sum_{d|W} k^2 |B|^{k-2} \varphi_W^2 \varphi_{W_1/d_1} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{2k+1}{p} \right) \sum_{j=0}^{\infty} 2^{-\exp(2^j/c k^2)} e^{(2k+1)2^{j+2}} \\
&= C_k k^2 |B|^{k-2} \varphi_W^2 \sum_{d|W} \varphi_{W_1/d_1} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{2k+1}{p} \right),
\end{aligned}$$

where

$$\begin{aligned}
C_k &= \sum_{j=0}^{\infty} 2^{-\exp(2^j/c k^2)} e^{(2k+1)2^{j+2}} \\
&= \sum_{j=0}^{\infty} e^{(2k+1)2^{j+2} - \exp(2^j/c k^2) \log 2} \\
&\leq \sum_{j=0}^{\infty} e^{(2k+1)2^{j+2} - \log 2 \left(\frac{2^j}{c k^2} + \frac{2^{2j}}{2c^2 k^4} \right)}.
\end{aligned}$$

The range of summation in j over $(-\infty, 0)$ can be omitted since $f(b_1, \dots, b_k) \geq \sum_{p \leq 2k} \gg \log \log k > 1$

for sufficiently large k . Now for $j \geq j_1 := \frac{\log(8c^2 k^4 (2k+1)/\log 2)}{\log 2}$, we have $\frac{2^{2j} \log 2}{2c^2 k^4} \geq (2k+1)2^{j+2}$ and $\frac{2^j}{c k^2} \geq 2^{j/2} \geq j$. Then

$$\sum_{j \geq j_1} e^{(2k+1)2^{j+2} - \log 2 \left(\frac{2^j}{c k^2} + \frac{2^{2j}}{2c^2 k^4} \right)} \leq \sum_{j \geq j_1} e^{-\log 2 \frac{2^j}{c k^2}} \leq \sum_{j \geq j_1} 2^{-j} \leq 1.$$

For $j \leq j_1$, the exponent $(2k+1)2^{j+2} - \exp(2^j/c k^2) \log 2$ is maximized when

$$2^j = c k^2 \log(4(2k+1) c k^2 / \log 2)$$

and

$$\begin{aligned}
&\sum_{j=0}^{j_1} e^{(2k+1)2^{j+2} - \exp(2^j/c k^2) \log 2} \\
&\leq \left(\frac{\log(8c^2 k^4 (2k+1)/\log 2)}{\log 2} \right) \cdot e^{2(2k+1) c k^2 (2 \log(2(2k+1) c k^2 / \log 2))}
\end{aligned}$$

Now we get

$$C_k \leq e^{c_5 k^3 \log k}$$

for some absolute constant $c_5 > 0$.

Now we turn back to S . The number of divisors d of W which gives the same d_2 is smaller than $\sum_{t=0}^{5k} \binom{5k}{t} = 2^{5k}$. And recall $|B| = \alpha\varphi_W$. Hence

$$\begin{aligned}
S &\leq C_k k^2 |B|^{k-2} \varphi_W^2 \sum_{d|W} \varphi_{W_1/d_1} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{2k+1}{p}\right) \\
&\leq C_k k^2 2^{5k} |B|^{k-2} \varphi_W^2 W \sum_{d_2|W_2} \frac{1}{d_2} \prod_{p|W_2/d_2} \left(1 - \frac{2k+1}{p}\right) \\
&\leq C_k^2 \alpha^{-2} |B|^k W \prod_{p|W_2} \left(1 - \frac{2k+1}{p}\right) \cdot \sum_{d_2|W_2} \frac{1}{d_2} \prod_{p|d_2} \left(1 - \frac{2k+1}{p}\right)^{-1} \\
&\leq C_k^2 \alpha^{-2} |B|^k W \prod_{p|W_2} \left(1 - \frac{2k+1}{p}\right) \left(1 + \frac{1}{p-2k-1}\right) \\
&\leq C_k^2 \alpha^{-2} |B|^k W \prod_{p|W_2} \left(1 - \frac{2}{p}\right)^k \\
&= C_k^2 \alpha^{-2} |B|^k \varphi_W^k W^{-k+1} \prod_{p|W_1} \left(1 - \frac{2}{p}\right)^{-k} \\
&\leq C_k^3 \alpha^{-2} |B|^k \varphi_W^k W^{-k+1} \\
&\leq \frac{e^{c_6 k^3 \log k} |B|^k \varphi_W^k}{\alpha^2 W^{k-1}}.
\end{aligned}$$

□

REFERENCES

- [1] J-R. Chen, *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica 16 (1973), 157-176.
- [2] K. Chipeniuk and M. Hamel, *On sums of sets of primes with positive relative density*, J. London Math. Soc. (2) 83 (2011) 673-690.
- [3] E. Croot, *Some properties of lower level-sets of convolutions*, preprint arXiv:1108.1578, 2011
- [4] Z. Cui, H. Li and B. Xue, *Long arithmetic progressions in $A + A + A$ with A a prime subset*, J of Number Theory, no.7, 132(2012), 1572-1582.
- [5] B. Green, *Roth's theorem in the primes*, Ann. of Math. 161(2005) 1609-1636.
- [6] B. Green and T. Tao, *Restriction theory of the Selberg sieve, with applications*, J. Theor. Nombres Bordeaux 18 (2006) 147-182
- [7] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. 167(2008) 481-547.
- [8] H. Iwaniec, *Sieve methods*. Graduate course, Rutgers 1996.
- [9] T. Lê, *Intersective polynomials and the primes*, J. Number Theory. 8(2010), 1705-1717.
- [10] H. Li and H. Pan, *Difference sets and polynomials of prime variables*, Acta. Arith. 138.1(2009), 25-52.
- [11] H. Li and H. Pan, *A density version of Vinogradov's three primes theorem*, Forum Mathematicum, no.4, 22(2010), 699-714.
- [12] K. Roth, *On certain sets of integers*, J. London Math. Soc. 28(1953), 104-109.
- [13] T. Tao and V. Vu, *Additive combinatorics* (Cambridge University Press, Cambridge, 2006).

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